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Three-dimensional subalgebras of simple Lie algebras: a *Mathematica*™ package

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Abstract
We propose a Mathematica package which allows us to determine all the classes of three-dimensional subalgebras (TDS) of a simple Lie algebra. The programming of the package is based on the theory of semisimple subalgebras of simple Lie algebras. We summarize the main points of this theory, which leads to an algorithm for the construction of the classes of TDS. A particular emphasis is laid on the construction of the exceptional TDS of D_n. The package `Decompositions.m` implements this algorithm to give all the TDS of the classical simple Lie algebras, principal and nonprincipal. The package provides several functions which characterize the three-dimensional embeddings such as, for example, the set of spins of a decomposition, the defining vector, the pi-system of roots, the generators of the TDS and the generators of each decomposition.

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1. Introduction

The classification of the semisimple subalgebras of the simple Lie algebras is of major importance for the study of the symmetries of the physical systems. In particular, the study of the embeddings of Lie subalgebras in algebras of a higher rank has become of great interest in the search of possibilities to extend the symmetries of some models to larger groups of symmetries.

In two-dimensional field theories and gravitation theories, the fundamental work of Zamolodchikov [1], Belavin and Polyakov [2] revealed the existence of additional symmetries, which extend the symmetry algebra of the theory, the Virasoro algebra. Hence, generalizations of these theories were introduced as higher spin extensions of the Virasoro algebra, known as W-algebras.

A systematic method to construct W-algebras is based on the so-called Hamiltonian reduction of Drinfeld–Sokolov [3]. In this approach, we can construct a W-algebra for any embedding of SL_2 in a Lie algebra. For a given Lie algebra there are several inequivalent embeddings. The first examples were the two W-algebras, associated with the two classes of three-dimensional subalgebras (TDS) of the Lie algebra SL_3. The standard W_n algebras correspond to the principal embedding of SL_2 in the Lie algebras SL_n and contain primary fields of conformal dimensions (or spins) $3, 4, \ldots, n$ and the energy–momentum tensor, which is a quasi-primary field, of conformal dimension 2. However, there are many other W-algebras which can also be obtained from any embedding of SL_2 in a Lie algebra.

It is then clear that the three-dimensional embeddings in a Lie algebra play a central role in the theory of W-algebras. Therefore, we propose in this paper a Mathematica package, which allows us to determine all the classes of TDS of a simple Lie algebra and to calculate the decompositions of simple Lie algebras with respect to SL_2 embeddings. An application of this package to the theory of W-gravity, with explicit examples of calculations, can be found in [4], where zero curvature equations are solved to obtain the anholomorphicity conditions for primary fields. The programming of this package is based on the theory of semisimple subalgebras of simple Lie algebras, introduced by Dynkin [5].

In section 2 we introduce some necessary basic elements and notation concerning the simple Lie algebras. We present the essential points of the theory of Dynkin, which leads to an algorithmic approach for the construction of all classes of three-dimensional embeddings in a Lie algebra. Particular emphasis is laid, in section 2.2.3, on the construction of the exceptional, nonprincipal TDS of the Lie algebras D_n.

In section 3 we present, in an algorithmic manner, the construction of the generators of a given Lie algebra, for each of its decompositions with respect to the TDS.

Section 4 is a user's guide for the Mathematica package `Decomposition.m`. For the aspects concerning the programming in Mathematica we refer to [6, 7].

Results have been obtained by direct calculus for Lie algebras of rank up to 4 in [8] and up to 6 in [9]. The package `Decomposition.m` recovers all these results and allows us to obtain new ones, for Lie algebras of a higher rank, the only limitations being the computer resources.

More precisely, the first part of the program, based on theoretical aspects presented in section 2, allows us to calculate all the pi-systems of a simple Lie algebra (maximal and nonmaximal), to determine the regular subalgebras associated with these pi-systems and to calculate all the defining vectors of the three-dimensional embeddings.

The second part of the program applies the theory presented in section 3, to determine the spin content of each decomposition, the generators and the structure constants of the Lie algebra, in each decomposition.

2. Subalgebras of a simple Lie algebra

We recall some elements of the classical simple Lie algebra theory, which are used in programming the Mathematica package `Decompositions.m`.

We consider a Lie algebra $\mathbf{g}$ of rank n and dimension d and a basis of this algebra which consists of n Cartan generators $\{H_1, \ldots, H_n\}$ and $d - n$ root generators $\{E_{\alpha_1}, \ldots, E_{\alpha_{d-n}}\}$, half of which are positive root generators and half are negative root generators, frequently denoted by $F_\alpha = E_{-\alpha}$.

The Cartan generators H_i span the maximal Abelian subalgebra $\mathbf{h}$ of $\mathbf{g}$. The subspaces spanned by root vectors E_α, denoted by $\mathbf{g}_\alpha$, are $\mathbf{h}$-invariant:

$$[H, E_\alpha] = \alpha(H)E_\alpha \qquad [H, F_\alpha] = -\alpha(H)F_\alpha \qquad \forall\ H \in \mathbf{h}.$$

In these relations, the applications $\alpha \in \mathbf{h}^*$ are the roots of the Lie algebra $\mathbf{g}$. We denote Σ the set of all roots (positive and negative) and Σ_0 the subset of simple roots which form a basis in $\mathbf{h}^*$.

On $\mathbf{g}$ is defined the symmetric bilinear Killing form,

$$K(x, y) = \mathrm{Tr}(\mathrm{ad}_x \cdot \mathrm{ad}_y) \qquad \forall\, x, y \in \mathbf{g} \tag{1}$$

where ad is the adjoint application, $\mathrm{ad}_x(y) = [x, y]$. This $\mathbf{g}$-invariant form is nondegenerate on $\mathbf{h}$ and defines, up to a constant c, a scalar product on $\mathbf{h}$:

$$\langle x, y\rangle = cK(x, y) \qquad \forall\, x, y \in \mathbf{h}.$$

This scalar product identifies the Cartan subalgebra $\mathbf{h}$ with its dual $\mathbf{h}^*$ by associating with each element $\alpha \in \mathbf{h}^*$ the element $H_\alpha \in \mathbf{h}$ such that $\alpha(H) = \langle H_\alpha, H\rangle$, for all $H \in \mathbf{h}$. We consider Cartan generators as elements of $\mathbf{h}$ which are associated with the simple roots.

The scalar product on $\mathbf{h}$ induces a scalar product on $\mathbf{h}^*$, defined as

$$\langle \alpha, \beta\rangle = \langle H_\alpha, H_\beta\rangle \qquad \forall\, \alpha, \beta \in \mathbf{h}^*. \tag{2}$$

These scalar products on $\mathbf{h}$ and $\mathbf{h}^*$ are determined by the Killing form, up to the constant c, which is fixed, as usual, by the condition that the square of the length of a long root is equal to 2. This normalization of the scalar product allows us to give an Euclidean representation of the roots. Hence, the scalar product on $\mathbf{h}^*$ is identified with the Euclidean scalar product. In the Mathematica package we use this identification to compute scalar products between roots.

We consider $\mathbf{h}_0^*$ the set of elements in $\mathbf{h}^*$ expressible as real linear combinations of the roots of $\mathbf{g}$, also called the idempotent of $\mathbf{g}$, which has the same dimension as the Cartan subalgebra [10]. We can define an order relation on $\mathbf{h}_0^*$, compatible with the vectorial space structure. Every such ordering determines a subsystem of positive roots and every system of simple roots determines an ordering on $\mathbf{h}_0^*$.

2.1. Regular subalgebras and S-subalgebras

To solve the problem of finding the subalgebras of a Lie algebra it is sufficient to determine all the maximal subalgebras. Then, the other subalgebras can be obtained in a recursive manner. In the first approach, all subalgebras can be separated into two disjoint classes: regular subalgebras and S-subalgebras. Hence, every subalgebra of a simple Lie algebra is either a regular subalgebra, an S-subalgebra, or an S-subalgebra of one of its proper regular subalgebras.

The theory developed by Dynkin to solve the problem of finding all the classes of TDS of a simple Lie algebra can be summarized in two essential points:

- It gives all the regular subalgebras of a simple Lie algebra
- It shows that all three-dimensional S-subalgebras of a simple Lie algebra are principal, up to a few exceptions, which occur in the cases of Lie algebras D_n, $E_{6,7,8}$.

We present hereafter the main definitions and results of this theory, which allows us to construct an algorithm for finding all TDS of a given simple Lie algebra.

Definition 2.1. *Let* $\mathbf{g}$ *be a simple Lie algebra and* $\tilde{\mathbf{g}} \subseteq \mathbf{g}$ *a Lie subalgebra, with the canonical decomposition:*

$$\tilde{\mathbf{g}} = \tilde{\mathbf{h}} \oplus \sum_{\tilde{\alpha} \in \tilde{\Sigma}} \tilde{\mathbf{g}}_{\tilde{\alpha}}.$$

The Lie subalgebra $\tilde{\mathbf{g}}$ *is called regular if there is a canonical decomposition of* $\mathbf{g}$*,*

$$\mathbf{g} = \mathbf{h} \oplus \sum_{\alpha\in\Sigma} \mathbf{g}_\alpha$$

such that the Cartan subalgebra $\tilde{\mathbf{h}} \subseteq \mathbf{h}$ *and the root system* $\tilde{\Sigma} \subseteq \Sigma$*.*

A S-subalgebra of $\mathbf{g}$ *is a subalgebra which is not contained in any proper regular subalgebra of* $\mathbf{g}$*.*

The previous definition is given in terms of the complete system of roots, but for our purpose it is more convenient to use the system of simple roots. According to the general properties of simple roots, if $\tilde{\alpha}$ and $\tilde{\beta}$ are simple roots of the subalgebra $\tilde{\mathbf{g}}$, then $\tilde{\alpha} - \tilde{\beta}$ is no longer a root of $\tilde{\mathbf{g}}$. On the other hand, $\tilde{\alpha} - \tilde{\beta}$ cannot be a root of $\mathbf{g}$, either. Otherwise, from the properties of subalgebras, the root vector corresponding to the root $\tilde{\alpha} - \tilde{\beta}$ should be an element of the subalgebra $\tilde{\mathbf{g}}$ and it is not since $\tilde{\alpha} - \tilde{\beta}$ is not a root of $\tilde{\mathbf{g}}$.

Therefore, the system of simple roots of a regular subalgebra is a particular subset of the root system Σ of $\mathbf{g}$, called a pi-system, which has the following properties.

Definition 2.2. *Let* Σ *be the root system of a Lie algebra* $\mathbf{g}$*. A subset* Π *of the root system* Σ *is called a pi-system if*

(1) for all $\alpha, \beta \in \Pi, \alpha - \beta \notin \Sigma$
(2) Π *is a linearly independent system.*

More precisely, theorem 5.1 of [5] shows that the regular subalgebras of $\mathbf{g}$ are in one to one correspondence with the pi-systems Π of Σ and are spanned by the generators $\{E_\alpha, F_\alpha, H_\alpha \,|\, \alpha \in \Pi\}$. Therefore, the problem of finding regular subalgebras of $\mathbf{g}$ comes down to the problem of finding all the pi-systems of the complete system of roots Σ of $\mathbf{g}$. Several results of [5] lead to an algorithmic approach to the problem of finding all the pi-systems for a given Lie algebra. This approach presents three steps:

(1) Firstly, we have to determine all the maximal pi-systems.
For a Lie algebra $\mathbf{g}$ of rank n, all the pi-systems with n elements can be obtained by an algorithmic construction, performing some *elementary transformations* on the system of simple roots of $\mathbf{g}$. An elementary transformation of a pi-system consists of three steps:
   (a) We start with a pi-system of simple roots. It determines an order relation on the root system Σ, with respect to which the elements of the pi-system are positive. Then, we add to the pi-system the smallest root of Σ with respect to this pi-system. This smallest root δ_m is defined as the linear combination of the elements of the pi-system, with negative integer coefficients, with the property $\delta_m - \alpha \notin \Sigma$, for all the roots α of the pi-system. The resulting system of roots, called an *extended system*, satisfies the condition (1) of definition 2.2, but not condition (2).
   The Dynkin diagram of the extended system is called an *extended diagram* and satisfies the construction rules for Dynkin diagrams.
   (b) For the extended system we remove arbitrarily one of the original elements. All the n subsets obtained in this way have n elements and also respect condition (2). Therefore, they are pi-systems.
   (c) The preceding procedure must be applied now to each one of the n diagrams with n-points obtained in step (b). These n-point diagrams can present several connecting parts. In this case, the procedure must be applied to each connecting part.
   This process is applied recursively, until no new pi-system is obtained.

Theorem 5.3 of Dynkin [5] shows that all the pi-systems with n elements can be obtained by these elementary transformations starting from a system of simple roots of $\mathbf{g}$.

(2) Secondly, we have to determine all the nonmaximal pi-systems.

It is obvious that every subset of a pi-system is also a pi-system and, from the theorem 5.2 of Dynkin [5], any pi-system with less then n elements can be extended to a maximal pi-system (with n elements). Therefore, for a Lie algebra of rank n, it is sufficient to find all the pi-systems with n elements and then to take all the subsets with r elements, $r < n$. In this way, we can construct all the pi-systems with r elements, which correspond to regular subalgebras of rank r.

(3) Finally, on the set of all pi-systems, obtained by iterative application of the first two steps, we must eliminate those which are Weyl equivalent, since they correspond to conjugate subalgebras.

Theorems 5.2 and 5.3 of [5] show that this algorithm gives all the possible pi-systems of simple roots for regular semisimple subalgebras of a simple Lie algebra, starting from its system of simple roots.

For the classical Lie algebras A_n, B_n, C_n, E_6, F_4, G_2, to each pi-system given by this algorithm corresponds one class of regular semisimple subalgebra. Exceptions occur in the case of regular semisimple subalgebras of D_n, E_7 and E_8, where for some particular pi-systems two classes of regular subalgebras correspond.

2.2. *Three-dimensional subalgebras*

All complex semisimple three-dimensional subalgebras are isomorphic with the Lie algebra SL_2, with the canonical generators: H, E, F and commutation relations:

$$[H, E] = 2E \tag{3}$$

$$[H, F] = -2F \tag{4}$$

$$[E, F] = H. \tag{5}$$

These subalgebras are characterized by the element $f \in \mathbf{h}^\star$, associated with H by the scalar product (2), called the *defining vector* of the three-dimensional subalgebra. Theorem 8.1 of [5] shows that two TDS of a semisimple algebra are conjugated iff their defining vectors are Weyl conjugate. In our package, we do not use this criterion of conjugacy, but another one, based on the characteristic of the TDS.

To define the *characteristic diagram*, let us consider f as the defining vector of a three-dimensional subalgebra of $\mathbf{g}$. This vector determines an order relation on the idempotent $\mathbf{h}_0^*$, if we consider positive, those roots of $\mathbf{g}$ which have positive scalar products with f. From these positive roots we choose a system of simple roots, with respect to this ordering, for the Lie algebra $\mathbf{g}$. With these simple roots we construct the Dynkin diagram of the Lie algebra $\mathbf{g}$ and we associate with each point of the diagram the scalar product of the defining vector with the corresponding simple root. The diagram with number labels on the points, obtained in this way, is called the characteristic diagram of the three-dimensional subalgebra in $\mathbf{g}$.

The labels which appear in a characteristic diagram can be one of the integers 0, 1 or 2. A three-dimensional subalgebra is called *principal* if all the labels of the corresponding characteristic diagram are equal to 2.

Theorem 8.2 [5] shows that two three-dimensional subalgebras of a semisimple algebra are conjugated iff their characteristics coincide.

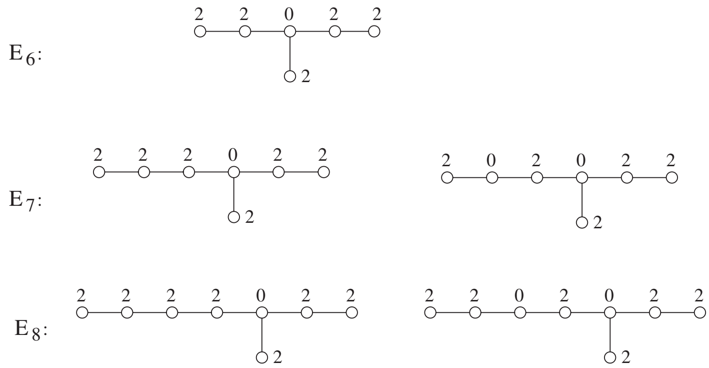

Figure 1. Characteristic diagrams of the nonprincipal TDS of E_6, E_7 and E_8.

The classification of all semisimple subalgebras of a simple algebra $\mathbf{g}$ shows that every three-dimensional subalgebra of $\mathbf{g}$ is either:

(a) an S-subalgebra of $\mathbf{g}$,
(b) an S-subalgebra of one of the proper regular subalgebras $\tilde{\mathbf{g}}$ of $\mathbf{g}$ or
(c) a regular subalgebra of $\mathbf{g}$.

Concerning case (a), the theorems 9.2 and 9.3 of [5] show that all three-dimensional S-subalgebras of A_n, B_n, C_n, F_4 and G_2 are principal.

For the Lie algebras D_n, beside the principal three-dimensional S-subalgebras, there are also $[(n-2)/2]$ classes of exceptional three-dimensional S-subalgebras, which are not principal (we denote by $[x]$ the integer part of x). For each $r = 1, \ldots, [(n-2)/2]$, we denote by D_n^{n-2r} the corresponding algebra of type D_n including the exceptional three-dimensional S-subalgebra. Its characteristic diagram is given in figure 2 and contains labels 0 at the positions $n-2r, n-2(r-1), \ldots, n-2$ and labels 2 elsewhere.

For the Lie algebra E_6, there is one exceptional class of three-dimensional S-subalgebras, besides the principal one, and for the Lie algebras E_7, E_8 there are two such exceptional classes, with characteristics given in figure 1.

In case (b), the three-dimensional S-subalgebra of a simple regular subalgebra $\tilde{\mathbf{g}}$ of $\mathbf{g}$ is also principal in $\tilde{\mathbf{g}}$, if $\tilde{\mathbf{g}}$ is of type A_n, B_n, C_n, F_4 or G_2. As in case (a), if the regular subalgebra is of type D_n or E, then, beside the principal TDS, there are also some exceptional TDS, which are not principal.

For the semisimple proper regular subalgebras, which decompose as the sum of simple ideals, $\tilde{\mathbf{g}} = \oplus \mathbf{g}_i$, we must find the three-dimensional subalgebras T_i of each simple ideal and their defining vectors f_i. Then, $T = \oplus T_i$ is a three-dimensional subalgebra of $\tilde{\mathbf{g}}$ and $f = \sum f_i$ is the corresponding defining vector.

Due to this result it is sufficient to describe the three-dimensional subalgebras (principal and nonprincipal) of the classical simple Lie algebras.

2.2.1. Principal three-dimensional subalgebras. Let $\mathbf{g}$ be one of the classical simple Lie algebras, Σ its system of roots and Π the system of simple roots of $\mathbf{g}$. For the algebra $\mathbf{g}$ we consider the canonical decomposition: $\mathbf{g} = \mathbf{h} \oplus \sum_{\alpha\in\Sigma} \mathbf{g}_\alpha$, where $\mathbf{h}$ is the Cartan subalgebra and $\mathbf{g}_\alpha$ are the root spaces of $\mathbf{g}$.

The construction of the generators H, E and F of the principal three-dimensional subalgebra of $\mathbf{g}$ is based on the following result.

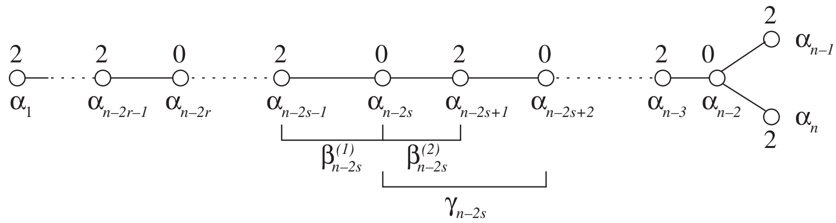

Figure 2. The diagram of the exceptional three-dimensional subalgebra of D_n^{n-2r}.

Theorem 2.3. *Let* $f \in \mathbf{h}^\star$ *with the property*

$$\langle f, \alpha_j \rangle = 2 \tag{6}$$

for all the simple roots $\alpha_j \in \Pi$ *of* $\mathbf{g}$. *Let* $H \in \mathbf{h}$ *be the Cartan element associated with* f *by the scalar product. Then, for any* $E \in \sum_{\alpha\in\Pi} \mathbf{g}_\alpha$ *and any* $F \in \sum_{\alpha\in\Pi} \mathbf{g}_{-\alpha}$, H *verifies the commutation relations (3) and (4) of* SL_2.

The element f of $\mathbf{h}^\star$, with the property (6), is the defining vector of the principal three-dimensional subalgebra of $\mathbf{g}$. The defining vector is written in the basis of simple roots as $f = \sum_{\alpha_i\in\Pi} x_i\alpha_i$, where the coefficients x_i are solutions of the system (6). Then the Cartan generator H of the principal TDS of $\mathbf{g}$ is

$$H = \sum_{\alpha_i\in\Pi} x_i H_{\alpha_i}.$$

The root generators E and F are linear combinations of root generators of simple roots and simple negative roots, respectively,

$$E = \sum_{\alpha_i\in\Pi} u_i E_{\alpha_i} \qquad F = \sum_{\alpha_i\in\Pi} v_i F_{\alpha_i}.$$

Since $\left[E_{\alpha_i}, F_{\alpha_j}\right] = \delta_{ij}\left\langle E_{\alpha_i}, F_{\alpha_j}\right\rangle H_{\alpha_i}$ for all simple roots $\alpha_i, \alpha_j \in \Pi$, equation (5) is verified if the coefficients u_i and v_i satisfy

$$u_i v_i \left\langle E_{\alpha_i}, F_{\alpha_i}\right\rangle = x_i \qquad \forall\ \alpha_i \in \Pi.$$

Hence, the coefficients u_i and v_i are not completely determined. In the package, we choose: $u_i = 1$ and $v_i = x_i \big/ \left\langle E_{\alpha_i}, F_{\alpha_i}\right\rangle$.

2.2.2. Nonprincipal three-dimensional subalgebras. The exceptional (nonprincipal) classes of three-dimensional subalgebras appear for the Lie algebras of type D_n and $E_{6,7,8}$. The defining vectors of these subalgebras are given, in the basis of simple roots, as $f = \sum_{\alpha_i\in\Pi} x_i\alpha_i$, where the coefficients x_i are solutions of the system obtained from the corresponding characteristic diagrams of figures 2 and 1, respectively. In these cases, theorem 4.2 of [5] shows that if

$$\Gamma = \{\alpha \in \Sigma, \langle \alpha, f\rangle = 2\} \tag{7}$$

then the defining vector of the nonprincipal TDS can be written as

$$f = \sum_{\alpha\in\Gamma} c_\alpha \alpha \tag{8}$$

with some coefficients c_α (not necessarily unique), determined by the conditions $\langle\alpha, f\rangle = 2$, $\forall\ \alpha \in \Gamma$. Then the root generators, defined by

$$E = \sum_{\alpha\in\Gamma} u_\alpha E_\alpha \qquad F = \sum_{\alpha\in\Gamma} v_\alpha F_\alpha \tag{9}$$

satisfy automatically the commutation relations (3), (4) and the commutator $[E, F]$ is an element of $\mathbf{h}$. The condition (5), that this commutator is exactly the Cartan generator H, has the following explicit form

$$\begin{aligned}\sum_{\alpha_i\in\Pi} x_i H_{\alpha_i} = \sum_{\alpha\in\Gamma} u_\alpha v_\alpha \langle E_\alpha, F_\alpha\rangle H_\alpha + \sum_{\alpha,\beta\in\mathcal{N},\alpha\neq\beta} u_\alpha v_\beta N_{\alpha,-\beta} E_{\alpha-\beta} \\ + \sum_{\alpha\in\Gamma_0,\beta\in\mathcal{N},\beta-\alpha\in\Sigma_+} N_{\alpha,-\beta}(u_\alpha v_\beta E_{-(\beta-\alpha)} + u_\beta v_\alpha E_{\beta-\alpha})\end{aligned} \tag{10}$$

where Σ_+ is the set of positive roots, $\Gamma_0 = \{\alpha \in \Pi, \langle\alpha, f\rangle = 2\}$ and $\mathcal{N} = \{\alpha \in \Sigma_+, \text{nonsimple}, \langle\alpha, f\rangle = 2\}$ are disjoint subsets of Γ and $N_{\alpha,-\beta}$ are structure coefficients of the Lie algebra, $[E_\alpha, F_\beta] = N_{\alpha,-\beta}E_{\alpha-\beta}$. This system should determine the coefficients u_α and v_α of the root generators (9). However, it is not clear which of these coefficients can be chosen arbitrarily and we do not have a general method to solve the system (10), which is nonlinear in u_α and v_α.

An explicit form of the generators E, F of type (9) is given in [5] for the nonprincipal TDS of the exceptional Lie algebras $E_{6,7,8}$. For the Lie algebras of type D_n, no explicit form of these root generators is available in the literature, to our knowledge. Therefore, we present in detail the construction of these generators, as they are considered in the Mathematica package.

2.2.3. Nonprincipal three-dimensional subalgebras of the Lie algebras D_n. The nonlinear system (10) leads to a linear one, with the unknown v_α if the coefficient u_α are prescribed. The problem is how to prescribe the coefficient u_α such that the remaining system can be solved. In this section we propose a solution of this problem in the case of the simple Lie algebra D_n.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the system of simple roots of the algebra D_n and f the defining vector of the three-dimensional S-subalgebra of D_n^{n-2r}, given by the characteristic diagram of figure 2. In this case, the set Γ defined by (7) splits into three disjoint subsets: $\Gamma = \Gamma_0 \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma$, where Γ_0 is the subset of simple roots of Γ, $\mathcal{N}_\beta$ and $\mathcal{N}_\gamma$ are the subsets of Γ consisting of nonsimple positive roots which can be written as the sum of two and three simple roots, respectively. Taking into account the structure of the roots of the Lie algebra D_n and the characteristic diagram of figure 2, these sets are

$$\begin{aligned}&\Gamma_0 = \{\alpha \in \Pi, \langle\alpha, f\rangle = 2\} = \{\alpha_j\}_{j=1,\ldots,n-2r-1} \cup \{\alpha_{n-2s+1}\}_{s=1,\ldots,r} \cup \{\alpha_n\}\\ &\mathcal{N}_\beta = \left\{\beta^{(1)}_{n-2s} = \alpha_{n-2s-1} + \alpha_{n-2s}, \beta^{(2)}_{n-2s} = \alpha_{n-2s} + \alpha_{n-2s+1}\right\}_{s=1,\ldots,r} \cup \left\{\beta^{(3)}_{n-2} = \alpha_{n-2} + \alpha_n\right\}\\ &\mathcal{N}_\gamma = \{\gamma_{n-2s} = \alpha_{n-2s} + \alpha_{n-2s+1} + \alpha_{n-2s+2}\}_{s=2,\ldots,r}.\end{aligned}$$

With this notation, the generators E and F have the following form:

$$E = \sum_{\alpha\in\Gamma_0} u_\alpha E_\alpha + \sum_{\beta\in\mathcal{N}_\beta} u_\beta E_\beta + \sum_{\gamma\in\mathcal{N}_\gamma} u_\gamma E_\gamma \qquad F = \sum_{\alpha\in\Gamma_0} v_\alpha F_\alpha + \sum_{\beta\in\mathcal{N}_\beta} v_\beta F_\beta + \sum_{\gamma\in\mathcal{N}_\gamma} v_\gamma F_\gamma.$$

To compute the commutator $[E, F]$ we take into account the following properties of the roots of D_n, which essentially result from the diagram of figure 2:

(1) For all $\alpha, \beta \in \Gamma$, if $\alpha - \beta$ is a root, then $\alpha - \beta \in J_0 = \{\alpha \in \Pi, \langle f, \alpha\rangle = 0\}$.
(2) If $\alpha \in \Gamma_0$ and $\beta \in \mathcal{N}_\gamma$, then $\beta - \alpha$ cannot be a root of D_n, since if $\beta - \alpha$ is a sum of two simple roots, then it is not simple and therefore it cannot be in J_0.

(3) If $\alpha, \beta \in \mathcal{N}_\beta$ or $\alpha, \beta \in \mathcal{N}_\gamma$, then $\alpha - \beta$ is not a root of D_n.
(4) The generators of D_n are normalized such that $\langle E_\alpha, F_\alpha \rangle = 1$ for all root α.

With these properties we obtain

$$[E, F] = \sum_{\alpha\in\Gamma} u_\alpha v_\alpha H_\alpha + \sum_{\substack{\alpha\in\Gamma_0,\beta\in\mathcal{N}_\beta \\ \beta-\alpha\in J_0}} N_{\alpha,-\beta}(u_\alpha v_\beta E_{-(\beta-\alpha)} + u_\beta v_\alpha E_{\beta-\alpha}) + \sum_{\substack{\beta\in\mathcal{N}_\beta,\gamma\in\mathcal{N}_\gamma \\ \gamma-\beta\in J_0}} N_{\beta,-\gamma}(u_\beta v_\gamma E_{-(\gamma-\beta)} + u_\gamma v_\beta E_{\gamma-\beta}). \tag{11}$$

The essential point in the construction of the generators E and F of the exceptional TDS of D_n is that we can put to zero all the coefficients u_β, $\beta \in \mathcal{N}_\beta$, with two exceptions, $u_{\beta^{(2)}_{n-2}}$ and $u_{\beta^{(3)}_{n-2}}$:

$$\begin{aligned} &u_{\beta^{(1)}_{n-2s}} = 0 \qquad \text{for} \quad s = 1, \dots, r \qquad \text{for} \quad u_{\beta^{(2)}_{n-2s}} = 0 \qquad \text{for} \quad s = 2, \dots, r \\ &u_{\beta^{(2)}_{n-2}} \neq 0 \qquad u_{\beta^{(3)}_{n-2}} \neq 0. \end{aligned} \tag{12}$$

Moreover, we can also fix to 1 the coefficients of type u_α and u_γ:

$$u_\alpha = u_\gamma = 1 \qquad \alpha \in \Gamma_0 \qquad \gamma \in \mathcal{N}_\gamma. \tag{13}$$

With this choice, which seems to us as simple as possible, the system $[E, F] = H$ is solvable and there is only one generator F which satisfies it. In the rest of this section we prove this assertion.

With the choice (12), (13), the root generator E becomes

$$E = \sum_{\alpha\in\Gamma_0} E_\alpha + u_{\beta^{(2)}_{n-2}} E_{\beta^{(2)}_{n-2}} + u_{\beta^{(3)}_{n-2}} E_{\beta^{(3)}_{n-2}} + \sum_{\gamma\in\mathcal{N}_\gamma} E_\gamma$$

and, taking into account the explicit form of the roots of D_n, the commutator (11) is

$$\begin{aligned} [E, F] = &\sum_{\alpha\in\Gamma_0\cup\mathcal{N}_\gamma} v_\alpha H_\alpha + u_{\beta^{(2)}_{n-2}} v_{\beta^{(2)}_{n-2}} H_{\beta^{(2)}_{n-2}} + u_{\beta^{(3)}_{n-2}} v_{\beta^{(3)}_{n-2}} H_{\beta^{(3)}_{n-2}} + E_{\alpha_{n-2r}} N_{\gamma_{n-2r},-\beta^{(1)}_{n-2r+2}} v_{\beta^{(1)}_{n-2r+2}} \\ &+ \sum_{s=2}^{r-1} E_{\alpha_{n-2s}} \left(N_{\beta^{(1)}_{n-2s+2},-\gamma_{n-2s}} v_{\beta^{(1)}_{n-2s+2}} + N_{\beta^{(2)}_{n-2s-2},-\gamma_{n-2s-2}} v_{\beta^{(2)}_{n-2s-2}} \right) \\ &+ E_{\alpha_{n-2}} \left(N_{\beta^{(2)}_{n-4},-\gamma_{n-4}} v_{\beta^{(2)}_{n-4}} + N_{\beta^{(2)}_{n-2},-\alpha_{n-1}} u_{\beta^{(2)}_{n-2}} v_{\alpha_{n-1}} + N_{\beta^{(3)}_{n-2},-\alpha_n} u_{\beta^{(3)}_{n-2}} v_{\alpha_n} \right) \\ &+ \sum_{s=2}^{r} F_{\alpha_{n-2s}} \left(N_{\beta^{(1)}_{n-2s},-\alpha_{n-2s-1}} v_{\beta^{(1)}_{n-2s}} + N_{\beta^{(2)}_{n-2s},-\alpha_{n-2s+1}} v_{\beta^{(2)}_{n-2s}} \right) \\ &+ F_{\alpha_{n-2}} \left(N_{\beta^{(1)}_{n-2},-\alpha_{n-3}} v_{\beta^{(1)}_{n-2}} + N_{\beta^{(2)}_{n-2},-\alpha_{n-1}} v_{\beta^{(2)}_{n-2}} + N_{\beta^{(3)}_{n-2},-\alpha_n} v_{\beta^{(3)}_{n-2}} \right). \end{aligned}$$

Therefore, the condition $[E, F] = H$ comes down to a system containing two types of equations:

$$u_\alpha v_\alpha = c_\alpha \qquad \forall\, \alpha \in \Gamma_0 \cup \mathcal{N}_\gamma \cup \left\{ \beta^{(2,3)}_{n-2} \right\} \tag{14}$$

where c_α are the coefficients (8) of the defining vector and the parameters u_α are fixed by (13), and

$$\begin{aligned} &N_{\beta^{(1)}_{n-2r+2},-\gamma_{n-2r}} v_{\beta^{(1)}_{n-2r+2}} = 0 \\ &N_{\beta^{(1)}_{n-2s+2},-\gamma_{n-2s}} v_{\beta^{(1)}_{n-2s+2}} + N_{\beta^{(2)}_{n-2s-2},-\gamma_{n-2s-2}} v_{\beta^{(2)}_{n-2s-2}} = 0 \qquad s = 2, \dots, r-1 \\ &N_{\beta^{(2)}_{n-4},-\gamma_{n-4}} v_{\beta^{(2)}_{n-4}} + N_{\beta^{(2)}_{n-2},-\alpha_{n-1}} u_{\beta^{(2)}_{n-2}} v_{\alpha_{n-1}} + N_{\beta^{(3)}_{n-2},-\alpha_n} u_{\beta^{(3)}_{n-2}} v_{\alpha_n} = 0 \\ &N_{\beta^{(1)}_{n-2s},-\alpha_{n-2s-1}} v_{\beta^{(1)}_{n-2s}} + N_{\beta^{(2)}_{n-2s},-\alpha_{n-2s+1}} v_{\beta^{(2)}_{n-2s}} = 0 \qquad s = 2, \dots, r \\ &N_{\beta^{(1)}_{n-2},-\alpha_{n-3}} v_{\beta^{(1)}_{n-2}} + N_{\beta^{(2)}_{n-2},-\alpha_{n-1}} v_{\beta^{(2)}_{n-2}} + N_{\beta^{(3)}_{n-2},-\alpha_n} v_{\beta^{(3)}_{n-2}} = 0 \end{aligned} \tag{15}$$

which comes from the condition that the coefficients of the root generators in $[E, F]$ are zero.

Part (14) of this system splits into two subsystems, which can be detailed as follows. The first one,

$$\begin{aligned}
&v_{\alpha_i} = x_i && i = 1, \ldots, n-2r-1 \\
&v_{\gamma_{n-2r}} = x_{n-2r} && \\
&v_{\gamma_{n-2s}} + v_{\alpha_{n-2s+1}} = x_{n-2s+1} && s = 3, \ldots, r \\
&v_{\gamma_{n-2s}} + v_{\gamma_{n-2s+2}} = x_{n-2s+2} && s = 3, \ldots, r \\
&v_{\gamma_{n-4}} + v_{\alpha_{n-3}} = x_{n-3} &&
\end{aligned}$$

decouples from the rest and can be solved to give v_{α_i}, with $i = 1, \ldots, n-2r-1$ and $v_{\alpha_{n-2s+1}}$ and $v_{\gamma_{n-2s}}$, with $s = 2, \ldots, r$. The second subsystem contains the last three equations of (14),

$$\begin{aligned}
&u_{\beta^{(2)}_{n-2}} v_{\beta^{(2)}_{n-2}} + u_{\beta^{(3)}_{n-2}} v_{\beta^{(3)}_{n-2}} = x_{n-2} - v_{\gamma_{n-4}} \\
&u_{\alpha_{n-1}} v_{\alpha_{n-1}} + u_{\beta^{(2)}_{n-2}} v_{\beta^{(2)}_{n-2}} = x_{n-1} \\
&u_{\alpha_n} v_{\alpha_n} + u_{\beta^{(3)}_{n-2}} v_{\beta^{(3)}_{n-2}} = x_n
\end{aligned} \tag{16}$$

which couples to the part (15) and must be solved together, with respect to the remaining unknowns $v_{\alpha_{n-1}}, v_{\alpha_n}, v_{\beta^{(1,2)}_{n-2s}}, s = 1, \ldots, r$ and $v_{\beta^{(3)}_{n-2}}$.

The first equation of (15) gives $v_{\beta^{(1)}_{n-2r+2}} = 0$. Recursive substitutions show that $v_{\beta^{(1,2)}_{n-2r+2}}, v_{\beta^{(1,2)}_{n-2r+6}}, \ldots, v_{\beta^{(1,2)}_{n-2r+2+4k}}, \ldots$ are all equal to zero. The remaining equations,

$$\begin{aligned}
&N_{\beta^{(2)}_{n-4}, -\gamma_{n-4}} v_{\beta^{(2)}_{n-4}} + N_{\beta^{(2)}_{n-2}, -\alpha_{n-1}} u_{\beta^{(2)}_{n-2}} v_{\alpha_{n-1}} + N_{\beta^{(3)}_{n-2}, -\alpha_n} u_{\beta^{(3)}_{n-2}} v_{\alpha_n} = 0 \\
&N_{\beta^{(1)}_{n-2}, -\alpha_{n-3}} v_{\beta^{(1)}_{n-2}} + N_{\beta^{(2)}_{n-2}, -\alpha_{n-1}} v_{\beta^{(2)}_{n-2}} + N_{\beta^{(3)}_{n-2}, -\alpha_n} v_{\beta^{(3)}_{n-2}} = 0 \\
&N_{\beta^{(1)}_{n-2s+2}, -\gamma_{n-2s}} v_{\beta^{(1)}_{n-2s+2}} + N_{\beta^{(2)}_{n-2s-2}, -\gamma_{n-2s-2}} v_{\beta^{(2)}_{n-2s-2}} = 0 \qquad s = r - (2k+1) \\
&N_{\beta^{(1)}_{n-2s}, -\alpha_{n-2s-1}} v_{\beta^{(1)}_{n-2s}} + N_{\beta^{(2)}_{n-2s}, -\alpha_{n-2s+1}} v_{\beta^{(2)}_{n-2s}} = 0 \qquad s = r - 2k
\end{aligned}$$

with $k = 0, 1, 2 \cdots$ and (16) form a system with unknowns $v_{\beta^{(1,2)}_{n-2r}}, v_{\beta^{(1,2)}_{n-2r+4}}, \ldots, v_{\beta^{(1,2)}_{n-2r+4k}}, \ldots, v_{\beta^{(2,3)}_{n-2}}, v_{\alpha_{n-1}}, v_{\alpha_n}$. The determinant of this remaining system is

$$\prod_{k=0}^{[\frac{r-3}{2}]} N_{\beta^{(1)}_{n-2r+4k}, -\alpha_{n-2r+4k-1}} N_{\beta^{(2)}_{n-2r+4k}, -\gamma_{n-2r+4k}} \Delta$$

where $\Delta = -N_{\beta^{(1)}_{n-2}, -\alpha_{n-3}} u_{\beta^{(2)}_{n-2}} u_{\beta^{(3)}_{n-2}} \left(N_{\beta^{(2)}_{n-2}, -\alpha_{n-1}} u_{\beta^{(2)}_{n-2}} - N_{\beta^{(3)}_{n-2}, -\alpha_n} u_{\beta^{(3)}_{n-2}} \right)$, if r is odd or $\Delta = N_{\beta^{(3)}_{n-2}, -\alpha_n} u_{\beta^{(2)}_{n-2}} - N_{\beta^{(2)}_{n-2}, -\alpha_{n-1}} u_{\beta^{(3)}_{n-2}}$, if r is even. Therefore, for an appropriate choice of $u_{\beta^{(2)}_{n-2}}$ and $u_{\beta^{(3)}_{n-2}}$ the system (14), (15) has only one solution. In the package, we choose $u_{\beta^{(2)}_{n-2}} = 1 / N_{\beta^{(2)}_{n-2}, -\alpha_{n-1}}$ and $u_{\beta^{(3)}_{n-2}} = 1 / \left(2 N_{\beta^{(3)}_{n-2}, -\alpha_n} \right)$ such that the determinant of the system is nonvanishing if r is odd, but also if r is even, due to the property $\left(N_{\beta^{(2)}_{n-2}, -\alpha_{n-1}} \right)^2 = \left(N_{\beta^{(3)}_{n-2}, -\alpha_n} \right)^2 = 1$ of the roots of D_n (see [10]).

3. Decompositions of a simple Lie algebra with respect to three-dimensional subalgebras

Decomposition of the Lie algebra **g** with respect to a three-dimensional subalgebra SL_2 usually refers to the decomposition of its adjoint representation as a direct sum of irreducible representations of SL_2. This decomposition is related to the eigenvalues and eigenspaces of the adjoint application ad_H of the Cartan element of SL_2.

Our goal in this section is to construct a basis for the Lie algebra consisting of eigenvectors of ad_H which have to satisfy additional constraints, similar to the commutation relations of the modes of primary fields. The basis obtained in this way is called the decomposition of **g** with respect to SL_2 and is of great interest in two-dimensional conformal field theories.

We consider first the three-dimensional subalgebra $\{L_{-1}, L_0, L_1\}$ of the Virasoro algebra, with the generators $\{L_n\}_{n\in\mathbb{Z}}$:

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{17}$$

This subalgebra is isomorphic with the Lie algebra SL_2 and we identify the generators $L_0 = -1/2H$, $L_1 = -E$, and $L_{-1} = F$. Then, the defining vector of this subalgebra is $f_0 = -1/2f$, the element associated with the generator L_0.

Then, we complete this basis to a basis of **g**, adding the so-called spin generators, W^s_m, which satisfy the commutation relations of the modes of primary fields:

$$\left[L_{-1}, W^s_m\right] = (-(s-1)-m)W^s_{m-1} \tag{18}$$

$$\left[L_0, W^s_m\right] = -mW^s_m \tag{19}$$

$$\left[L_1, W^s_m\right] = ((s-1)-m)W^s_{m+1}. \tag{20}$$

These generators are labelled by two indices: a spin index s and a spin projection index $m = -(s-1), \ldots, s-1$. The construction of the spin generators presents several steps.

Step 1. First we must determine the spins of the decomposition. Equation (19) shows that the values of the spin projections are the eigenvalues of the operator ad_{L_0}. These eigenvalues can be calculated directly. An optimal alternative is to use the defining vector f_0 of the three-dimensional subalgebra. Then, the eigenvalues of the operator ad_{L_0} are given by the scalar products $\langle f_0, \alpha\rangle$ of this defining vector with all the roots $\alpha \in \Sigma$ of **g**, to which we must add the eigenvalue 0, with the multiplicity equal to the rank of **g**.

Hence, in this set of eigenvalues we must identify all the spins of the decomposition, each one with its sequence of admissible spin projections. We denote by $\mathcal{S}$ the set of spins of a decomposition. In this set, we always consider the first element to be the spin 2 of the three-dimensional subalgebra and the others are put in decreasing order.

Step 2. For each spin s we determine the generator with minimal spin projection: $W^s_{-(s-1)}$. In this case, equations (18) and (19):

$$\left[L_{-1}, W^s_{-(s-1)}\right] = 0 \tag{21}$$

$$\left[L_0, W^s_{-(s-1)}\right] = (s-1)W^s_{-(s-1)} \tag{22}$$

show that the generator $W^s_{-(s-1)}$ is an element of $\mathrm{Ker}\,(\mathrm{ad}_{L_{-1}}) \bigcap V_{s-1}$, where $V_{s-1} = \sum_{\alpha\in\Gamma_{s-1}} \mathbf{g}_\alpha$ is the sum of the invariant root subspaces, corresponding to the roots:

$$\Gamma_{s-1} = \{\alpha \in \Sigma \mid \langle \alpha, f_0\rangle = s-1\}.$$

The space $\mathrm{Ker}\,(\mathrm{ad}_{L_{-1}}) \bigcap V_{s-1}$ is one-dimensional only for those spins which are not multiple. However, there are many decompositions of Lie algebras which present several spins with the same value, such as for example, the second decomposition of SL_3, which has the spins $\{2, 3/2, 3/2, 1\}$. In these cases the dimension of the space $\mathrm{Ker}\,(\mathrm{ad}_{L_{-1}}) \bigcap V_{s-1}$ is equal to the multiplicity of the spin s. The generators of minimal spin projection, $W^s_{-(s-1)}$, are taken to form a basis of this space.

To avoid confusions in the notation of the generators W, which can occur in the cases of multiple spins, we have added, as an upper index, the position i of the spin in the set $\mathcal{S}$ of spin values: $W_m^{s\,(i)}$.

Step 3. Finally, successive applications of equation (20) allows us to determine all the generators W_m^s with $m = -s+2, \ldots, s-1$, for each spin s. Note that the conditions (18) and (19) are automatically satisfied, at each iteration, due to the Jacobi identities.

4. The Mathematica package `Decompositions.m`

The theoretical aspects concerning the three-dimensional subalgebras of the classical simple Lie algebras, discussed previously, are implemented in the Mathematica package `Decompositions.m`. In this section we present this package and we give some guide lines for the users.

This package uses another Mathematica package [4] `SimpleLieAlgebras.m` and therefore, both have to be placed in the same directory, called `LieAlgebras`. These packages are available as a zipped archive `LieAlgebras.zip`, by e-mail from the authors or at *http://www.cpt.univ-mrs.fr/~garajeu*. The archive has to be decompressed into the home directory or into the subdirectory `AddOns/Applications` of the standard Mathematica distribution.

The package has to be loaded before using any of its symbols:

In[1]:= `<<LieAlgebras ‘Decompositions’`

Loading the package a second time will clear all previous definitions of symbols and all stored intermediate results. One can have the list of all symbols and functions provided by the package, using: `Names[‘‘LieAlgebras‘Decompositions‘*’’]`, as well as a description of each function, using `?nameoffunction`.

After loading the package we have first to give precisely which Lie algebra we are working with:

In[2]:= `SetLieAlgebra[’’type’’,n]`

This command fixes the settings of the problem, namely: ‘type’ is a string giving the type of the Lie algebra and `n` an integer giving the rank of the Lie algebra. This command calls a function of the package `SimpleLieAlgebras.m`, which initializes the Cartan–Weyl generators and the structure coefficients of the Lie algebra.

At this stage all the symbols and functions provided by the package are available, as well as those of the package `SimpleLieAlgebras.m`.

The package is thought up in two parts. The first one deals with theoretical aspects of section 2, to establish all the possible three-dimensional embeddings for the chosen Lie algebra. The second part follows section 3, to determine the structure of the Lie algebra decompositions.

4.1. Regular subalgebras and three-dimensional embeddings

Two main functions were defined to implement the algorithmic procedure described in section 2.1, for the construction of all TDS for the selected Lie algebra.

`MaximalPiSystem` applies step 1 of the algorithm to construct all the maximal pi-systems and the corresponding maximal regular subalgebras.

The function `MaximalPiSystem[options]` returns the list of all maximal pi-systems for the selected Lie algebra. Each pi-system is given as a matrix, having on the rows the

Euclidean coordinates of the roots. If the option `Diagrams->True` is given, this function gives a list of the maximal regular subalgebras associated with the maximal pi-systems and it draws their Dynkin diagrams.

In particular, this function gives the maximal pi-systems, which in table III of [9] are listed for Lie algebras up to rank 6, only.

`AllDecompositions` determines all the nonmaximal pi-systems, applying step 2 of the algorithm discussed in section 2.1 and all the TDS of the Lie algebra.

It prints the list of all decompositions and, for each of them, the following information: the defining vector (Euclidean components), the spins, the roots which form the pi-system and the regular subalgebras containing the three-dimensional subalgebra.

The function `AllDecompositions` also gives the list `DecompositionsList` of all decompositions of the Lie algebra. Each *i*th component of this list has six components which characterize the *i*th decomposition:

(1) Euclidean coefficients of the defining vector of the *i*th decomposition,
(2) spins of the *i*th decomposition,
(3) roots which form the pi-system of the *i*th decomposition,
(4) list of the equivalent pi-systems,
(5) equivalent defining vectors corresponding to the equivalent pi-systems,
(6) labels attached to the characteristic diagram of the *i*th decomposition.

Several additional functions were necessary to realize this algorithmic procedure.

`ListofSpins[defvec]` gives the list of spins of a decomposition characterized by a defining vector `defvec`. In this list, the values of the spins are arranged in decreasing order except the first one, which is fixed to be the spin 2 of the three-dimensional subalgebra. The variable `defvec` is the defining vector of the TDS, in Euclidean representation.

`MinRoot[ps]` computes the minimal root of the root system of the Lie algebra, with respect to the pi-system `ps`, as defined in step 1(a) of section 2.1. The matrix variable `ps` is a pi-system of roots, in Euclidean form.

`SimpleRoots[defvec]` selects, in the root system of the Lie algebra, the set of positive roots with respect to the vector `defvec` and gives those which are simple. The result is a list of these simple roots, in Euclidean form.

`DecompDefVector[i,options]` gives the defining vector for the *i*th decomposition, as a list of its coefficients in the basis of simple roots of the Lie algebra. This function allows us to compare our results with those of [9], table VI. If the option `EuclidForm->True` is chosen, the defining vector is given in Euclidean form.

`DefiningVectorPisystem[ps]` gives the list of the Euclidean representations of the defining vectors associated with a pi-system `ps`. This list contains one defining vector, given by (6), of the principal TDS of a Lie algebra which has `ps` as a system of simple roots. If `ps` is a system of simple roots for a Lie algebra of type D, then this list contains several defining vectors: one for the principal and one for each nonprincipal TDS. The variable `ps` is a matrix, which has on the rows, the Euclidean coordinates of the roots which form the pi-system `ps`.

`ConnexPart[ps]` gives the list of the connex parts of the Dynkin diagram of a pi-system `ps`.

`OrderedRoots[cps]` gives the list of the roots of a connecting part `cps` of a pi-system, in Euclidean representation, ordered as they appear in the Dynkin diagram.

Several graphical functions are also available. They are useful to compare our results with those of [5], tables 13, 16, 17.

`CharactDiagram[i]`, which gives the characteristic diagram of the *i*th decomposition.

`ShowDiagram[ps,listlabel]`, which draw the Dynkin diagram of the regular subalgebra, having the pi-system `ps` as a system of simple roots. The argument `listlabel` is optional and contains the labels attached to the diagram.

4.2. The structure of a Lie algebra decomposition with respect to SL_2 embeddings

`SetDecomposition[i]` fixes the symbols DefVect, Spins, PiSystem to the corresponding values of the *i*th decomposition of the Lie algebra given by the function `AllDecompositions`. It also computes the generators and the structure coefficients of the *i*th decomposition.

Once the decomposition *i* is fixed, the following functions, concerning this decomposition, are available.

`Spins` is the list of spins of the decomposition. In this list, the values of the spins are arranged in decreasing order except the first one, which is fixed to be the spin 2 of the three-dimensional subalgebra.

`DefVect` is the defining vector of the decomposition, given as a list of its Euclidean coordinates.

`PiSystem` is the pi-system of the decomposition. It is given as a matrix, having on the rows the Euclidean coordinates of the roots of the pi-system.

`W[k,s,m,options]` gives the generators $W_m^{s\,(k)}$ of the chosen decomposition, as a list of their coefficients with respect to the Cartan–Weyl basis. $W_m^{s\,(k)}$ is the generator of spin s and spin projection m and k is the position of the spin of value s in the list of the spins of the decomposition. This variable serves to distinguish between generators with the same value of spin, in the case of decompositions with multiple spins. If the option `BasisElement->True` is chosen, these generators are given in matrix form.

`PrintWGenerators[options]` prints the generators of the three-dimensional decomposition in matrix form. If the option `InternalForm->True` is chosen, these generators are given as lists of their coefficients with respect to the Cartan–Weyl basis.

`DecompGenList` prints the list of generators of the decomposition.

`DecompStr` is a three-level list containing the structure coefficients of the three-dimensional decomposition. This symbol is computed by the internal function DecompStrConst, called by the function SetDecomposition.

`DecompCommutationTable[]` prints the commutation table of the Lie algebra in the chosen decomposition. `DecompCommutationTable[{l1,l2},{c1,c2}]` prints only the part of the commutation table between rows l1 and l2 and columns c1 and c2.

Some of these elements, which characterize the decompositions of the Lie algebra, can also by calculated directly, without fixing the decomposition with `SetDecomposition`.

`DecompLGen[i,k]` gives the three Virasoro generators L_k of the three-dimensional subalgebra, for the *i*th decomposition. They are given as a list of their coefficients with respect to the Cartan–Weyl basis. The values of $k = -1, 0, 1$ correspond to the generators L_{-1}, L_0 and L_1, respectively.

`DecompGenerators[i]` computes the generators `W[k,s,m]`, of spin s and spin projection m, for the *i*th decomposition of the Lie algebra. They are given as a list of coefficients with respect to the Cartan–Weyl basis.

In the package `Decompositions.m`, some other more general functions, which could be useful, are also defined, as internal functions. For example, the function `SpecialInverse[n,m]` is an optimized function which gives the inverse of a particular square matrix `m`, which can be put in a three-block diagonal form by the function `ToBlockMatrix[n,m]`.

5. Conclusions

In this paper we have presented a Mathematica package which performs the decompositions of classical simple Lie algebras with respect to SL_2 subalgebras. The package allows us to determine the regular subalgebras and the classes of three-dimensional subalgebras (principal and nonprincipal) of all simple Lie algebras of types A, B, C, D, F_4 and G_2.

The user can access the structure of any Lie algebra decomposition (generators, structure coefficients, spins, etc) and can use these elements in his own calculations.

This package could still be developed in order to eliminate some of the limitations of the current version. For Lie algebras of high dimensions, several optimizations of the program could be necessary, concerning the memory usage and the time of calculations.

The program has important applications in several branches of mathematics and theoretical physics, such as for example in extended two-dimensional conformal field theories.

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